International Journal of Emerging Engineering Research and Technology
Volume 2, Issue 5, August 2014, PP 89-98
ISSN 2349-4395 (Print) & ISSN 2349-4409 (Online)

Free Vibration Analysis of Rectangular Plates Using a New Triangular Shear Flexible Finite Element

Mihir Chandra Manna

Department of Aerospace Engineering and Applied Mechanics
Indian Institute of Engineering Science and Technology,
Shibpur, Howrah, India
mcmbecdu@gmail.com

Abstract: *In this paper, free vibration analyses of isotropic and composite rectangular plates with different thickness ratios, aspect ratios, boundary conditions, and stacking sequences as applicable have been investigated using a new triangular shear flexible finite element. First-order shear deformation theory (FOSDT) is used to include the effect of transverse shear deformation. The element has six nodes on the sides and forty five degrees of freedom .The geometry of the element is expressed by shape function in terms of area coordinates. The finite element formulation is displacement-based. Numerical examples are presented to show the accuracy and elegancy of the proposed element. The non-dimensional frequency parameters of isotropic and composite plates under different thickness ratios, aspect ratios etc., are presented in tabular form along with the available published results.*

Keywords: *Rectangular plates, FOSDT, transverse shear, degrees of freedom.*

1. Introduction

Free vibration analysis of plate is very important in the field of structural engineering because of its wide application in practical life. The finite element method[1] is regarded as one of the most versatile analysis tools specifically in structural mechanics problems. The analysis of plates and shells are the first problems where the finite element method was first applied. Kirchoff's hypothesis was utilized where a number of problems were faced. The major problem concerned the satisfaction of normal slope continuity at the element edges which could not be solved satisfactorily. In the subsequent study, the above problem has been avoided by adopting Mindlin's hypothesis where the effect of shear deformation has been considered.

Many exact solutions for elastic thin isotropic plate for bending have been well documented in Timoshenko's monographs[2,3]. However, the analytical solution of plates with higher thickness ratio is not sufficient in literatures. Analysis of rectangular and triangular plates with different thickness ratios is quite vast. But there are limited numbers of literatures on triangular plates particularly for higher thickness ratios. Some papers have been discussed below which related to present work.

Zienkiewiczet al.[4]have proposed a discrete shear triangular element which is formulated based on Discrete Kirchhoff Theory. Chen et al.[5] proposed two refined triangular thin/thick plate elements, the conforming displacement element DKTM with one point quadrature for the part of shear strain and the element RDKTM with the re-constitution of the shear strain, based on the Mindlin/Reissner plate theory. In the formulations the exact displacement function of the Timoshenko's beam is used to derive the element displacements of the refined elements.Thankamet al.[6] described the buckling behavior of rectangular laminate plates subject to thermal loads. The element is based on the transverse displacement field.Gauss Quadrature formula is used to compute element matrices. Shufrinet al.[7] have been investigated the free vibration of rectangular thick plates with variable thickness and different boundary conditions by using the extended Kantorovich method. Two shear deformation theories which included the effect of both transverse shear stresses and rotary inertia have been applied to the analysis. Sang Wook Kang et al.[8] proposed a practical analytical method for the free vibration analysis of a simply supported rectangular plate with unidirectional arbitrary thickness variation. Huang et al.[9] presented a method which is developed for analyzing the free vibration problem of orthotropic rectangular plates with variable thickness.

Cai et al.[10]proposed a new three node triangular plate element, labeled as DST-S6 (Discrete Shear Triangular element with 6 extra Shear degrees of freedom), for the analyses of plate/shell structures comprising of thin or thick members. The formulation is based on the DKT (Discrete Kirchhoff Technique) and an appropriate use of the independent shear DOF.

Liewet. Al.[11] presented the comprehensive sets of accurate vibration frequencies for thick rectangular plates subjected to 21 boundary conditions involving all possible combination of clamped, simply supported and free edges. In this study, sets of mathematically complete two-dimensional polynomials are assumed in the displacement and rotational functions to approximate mode shapes. The energy function has been derived using the Rayleigh-Ritz procedure which leads to the governing eigen value equations. Sets of reasonably accurate vibration frequencies are presented for a wide range of aspect ratios a/b and relative thickness t/b for each boundary condition.

Liew[12] presented an analysis of free flexural vibration of thick symmetric rectangular laminates in which the Ritz method with a set of admissible beam characteristic orthogonal polynomials is used. A simple first order Reissner/Mindlin shear deformation theory was employed to account for the transverse shear effects.

Choo et al.[13]developed two plate bending element, one with 9 DOF triangular and other one with 12 DOF quadrilateral based on the hybrid-Trefftz method. Among the two independent displacement fields, i.e. the internal and the boundary displacements, they used the Mindlin–Reissner's thick plate solution with a particular solution under pressure load as the internal displacement field. Boundary displacement fields are approximated as transverse displacement $\widetilde{w}$and rotations$\widetilde{\beta_x}$ and $\widetilde{\beta_y}$ by cubic and quadratic hierarchical shape functions, respectively. Transverse shear strains are derived from constitutive equations and equilibrium equations, respectively, and additional degrees of freedom of hierarchical shape functions are removed using the relations between these two shear strains.

Brasile[14]presented a new assumed stress triangular element for Reissner–Mindlin plates, called TIP3, with three nodes and three degrees of freedom per node. The kinematics is constructed by means of the so-called linked interpolation ruled by technically significant degrees of freedom (i.e. one transversal displacement and two rotations per node) without using additional bubble modes. The static representation starts from a moment–shear uncoupled polynomial approximation and is constrained to satisfy some equilibrium conditions in order to reduce the stress parameters to a minimum number.

Zhuang et al.[15] have developed a new locking-free triangular thick plate element with 9 standard kinematic degrees of freedom and 6 additional degrees of freedom for shear strains for analyzing plate/shell structures of thin or thick members.

In the present work, free vibration of isotropic and composite rectangular plates with different thickness ratios, aspect ratios, boundary conditions, and stacking sequences has been analyzed using a new triangular plate element. To include the effect of transverse shear deformation, first-order shear deformation theory (FOSDT) is used. The element has six nodes on the sides and forty five degrees of freedom. Numerical examples are solved to show the accuracy and elegancy of the proposed element. The non-dimensional frequency parameters of isotropic and composite plates under different thickness ratios, aspect ratios etc., are presented in tabular form along with the available published results.

2. FINITE ELEMENT FORMULATION

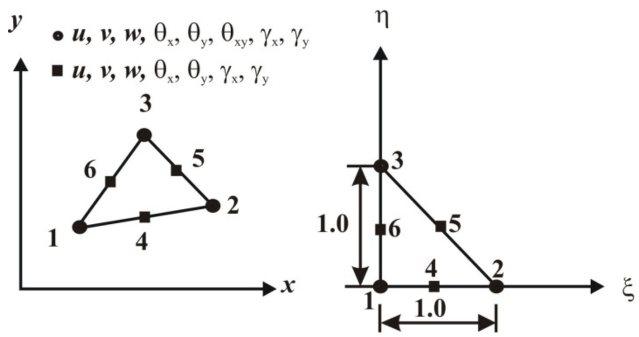

Fig. 1. *Element configuration*

The formulation is based on the Reissner-Mindlin plate theory. In this theory it is assumed that the transverse deflection of the plate is small compared to the plate thickness and the normal to the plate mid surface which is taken as the reference plane remains straight but may not remain normal to the deformed mid surface. The proposed element is shown in Fig.1.

The element has eight degrees of freedom $(u, v, w, \theta_x, \theta_y, \theta_{xy}, \gamma_x, \gamma_y)$ at nodes 1-3 and seven degrees of freedom $(u, v, w, \theta_x, \theta_y, \gamma_x, \gamma_y)$ at nodes 4-6. Nodes 1-3 are at the vertices and 4-6 at midpoints of the sides of the element. The natural co-ordinates of the nodes are $(0,0)$, $(1,0)$, $(0,1)$, $(0.5,0)$, $(0.5,0.5)$ and $(0,0.5)$. The co-ordinates of any point within the element with respect to the global co-ordinate system are given by

$$x = (1-\xi-\eta)x_1 + \xi x_2 + \eta x_3,\ y = (1-\xi-\eta)y_1 + \xi y_2 + \eta y_3,\ \xi = \frac{a_2 + b_2 x + c_2 y}{2\Delta},\ \eta = \frac{a_3 + b_3 x + c_3 y}{2\Delta} \tag{1}$$

where

$a_1 = x_2 y_3 - x_3 y_2$, $a_2 = x_3 y_1 - x_1 y_3$, $a_3 = x_1 y_2 - x_2 y_1$,

$b_1 = y_2 - y_3$, $b_2 = y_3 - y_1$, $b_3 = y_1 - y_2$, $c_1 = x_3 - x_2$,

$c_2 = x_1 - x_3$, $c_3 = x_2 - x_1$ and $\Delta = (a_1 + a_2 + a_3)/2$.

The field variables i.e., the independent displacement components at the reference plane may be expressed as follows:

$$u=[N_1]\{\alpha_u\},\ v=[N_1]\{\alpha_v\},\ w=[N_2]\{\alpha_w\},\ \gamma_x=[N_1]\{\alpha_{\gamma_x}\},$$

$$\text{and } \gamma_y=[N_1]\{\alpha_{\gamma_y}\} \tag{2}$$

where

$$\{\alpha_u\}=[\alpha_1\ \ \alpha_2\ \ \alpha_3\ \ \alpha_4\ \ \alpha_5\ \ \alpha_6]^T,\ \{\alpha_v\}=[\alpha_7\ \ \alpha_8\ \ \alpha_9\ \ \alpha_{10}\ \ \alpha_{11}\ \ \alpha_{12}]^T,$$

$$\{\alpha_w\}=[\alpha_{13}\ \ \alpha_{14}\ \ \alpha_{15}\ \ \alpha_{16}\ \ \alpha_{17}\ \ \alpha_{18}\ \ .\ .\ .\ \ \alpha_{33}]^T,$$

$$\{\alpha_{\gamma_x}\}=[\alpha_{34}\ \ \alpha_{35}\ \ \alpha_{36}\ \ \alpha_{37}\ \ \alpha_{38}\ \ \alpha_{39}]^T,\ \{\alpha_{\gamma_y}\}=[\alpha_{40}\ \ \alpha_{41}\ \ \alpha_{42}\ \ \alpha_{43}\ \ \alpha_{44}\ \ \alpha_{45}]^T,$$

$[N_1]=\left[1\ \ \xi\ \ \eta\ \ \xi^2\ \ \xi\eta\ \ \eta^2\right]$ and

$$[N_2] = [1\ \ \xi\ \ \eta\ \ \xi^2\ \ \xi\eta\ \ \eta^2\ \ \xi^3\ \ \xi^2\eta\ \ \xi\eta^2\ \ \eta^3\ \ \xi^4$$
$$\xi^3\eta\ \ \xi^2\eta^2\ \ \xi\eta^3\ \ \eta^4\ \ \xi^5\ \ \xi^4\eta\ \ \xi^3\eta^2\ \ \xi^2\eta^3$$
$$\xi\eta^4\ \ \eta^5].$$

Using Eqs. (1) and (2), θ_x, and θ_y may be written as

$$\begin{aligned}\theta_x &= \frac{\partial w}{\partial x} + \gamma_x = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x} + [N_1]\{\alpha_{\gamma_x}\} \\ &= \left(\frac{b_2}{2\Delta}\frac{\partial}{\partial \xi}[N_2] + \frac{b_3}{2\Delta}\frac{\partial}{\partial \eta}[N_2]\right)\{\alpha_w\} + [N_1]\{\alpha_{\gamma_x}\} \\ &= [N_3]\{\alpha_w\} + + [N_1]\{\alpha_{\gamma_x}\},\end{aligned} \tag{3}$$

$$\theta_y = \frac{\partial w}{\partial y} + \gamma_y = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y} + [N_1]\{\alpha_{\gamma_y}\}$$
$$= \left(\frac{c_2}{2\Delta}\frac{\partial}{\partial \xi}[N_2] + \frac{c_3}{2\Delta}\frac{\partial}{\partial \eta}[N_2]\right)\{\alpha_w\} + [N_1]\{\alpha_{\gamma_y}\} \quad (4)$$
$$= [N_4]\{\alpha_w\} + [N_1]\{\alpha_{\gamma_y}\},$$

Again, using Eqs. (3) and(4), θ_{xy} may be expressed as

$$\theta_{xy} = \frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y} = \left(\frac{\partial \theta_y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \theta_y}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \theta_x}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \theta_x}{\partial \eta}\frac{\partial \eta}{\partial y}\right)$$
$$= \left(\frac{b_2c_2}{2\Delta^2}\frac{\partial^2}{\partial \xi^2}[N_2] + \frac{b_3c_2+b_2c_3}{2\Delta^2}\frac{\partial^2}{\partial \xi \partial \eta}[N_2] + \frac{b_3c_3}{2\Delta^2}\frac{\partial^2}{\partial \eta^2}[N_2]\right)\{\alpha_w\}$$
$$+\left(\frac{c_2}{2\Delta}\frac{\partial}{\partial \xi}[N_1] + \frac{c_3}{2\Delta}\frac{\partial}{\partial \eta}[N_1]\right)\{\alpha_{\gamma_x}\} + \left(\frac{b_2}{2\Delta}\frac{\partial}{\partial \xi}[N_1] + \frac{b_3}{2\Delta}\frac{\partial}{\partial \eta}[N_1]\right)\{\alpha_{\gamma_y}\} \quad (5)$$
$$= [N_5]\{\alpha_w\} + [N_6]\{\alpha_{\gamma_x}\} + [N_7]\{\alpha_{\gamma_y}\}, \quad (5)$$

where

$$[N_3] = \frac{b_2}{2\Delta}\frac{\partial}{\partial \xi}[N_2] + \frac{b_3}{2\Delta}\frac{\partial}{\partial \eta}[N_2],\ [N_4] = \frac{c_2}{2\Delta}\frac{\partial}{\partial \xi}[N_2] + \frac{c_3}{2\Delta}\frac{\partial}{\partial \eta}[N_2],$$

$$[N_5] = \frac{b_2c_2}{2\Delta^2}\frac{\partial^2}{\partial \xi^2}[N_2] + \frac{b_3c_2+b_2c_3}{2\Delta^2}\frac{\partial^2}{\partial \xi \partial \eta}[N_2] + \frac{b_3c_3}{2\Delta^2}\frac{\partial^2}{\partial \eta^2}[N_2],$$

$$[N_6] = \frac{c_2}{2\Delta}\frac{\partial}{\partial \xi}[N_1] + \frac{c_3}{2\Delta}\frac{\partial}{\partial \eta}[N_1] \text{ and } [N_7] = \frac{b_2}{2\Delta}\frac{\partial}{\partial \xi}[N_1] + \frac{b_3}{2\Delta}\frac{\partial}{\partial \eta}[N_1].$$

Eqs. (2) – (5) may be assembled in matrix form as

$$\begin{Bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} = \begin{bmatrix} [N_1] & [N_0] & [N_{00}] & [N_0] & [N_0] \\ [N_0] & [N_1] & [N_{00}] & [N_0] & [N_0] \\ [N_0] & [N_0] & [N_2] & [N_0] & [N_0] \\ [N_0] & [N_0] & [N_3] & [N_1] & [N_0] \\ [N_0] & [N_0] & [N_4] & [N_0] & [N_1] \\ [N_0] & [N_0] & [N_5] & [N_6] & [N_7] \\ [N_0] & [N_0] & [N_{00}] & [N_1] & [N_0] \\ [N_0] & [N_0] & [N_{00}] & [N_0] & [N_1] \end{bmatrix} \begin{Bmatrix} \alpha_u \\ \alpha_v \\ \alpha_w \\ \alpha_{\gamma_x} \\ \alpha_{\gamma_y} \end{Bmatrix} = [N_A] \begin{Bmatrix} \alpha_u \\ \alpha_v \\ \alpha_w \\ \alpha_{\gamma_x} \\ \alpha_{\gamma_y} \end{Bmatrix} \text{ for nodes } 1-3 \quad (6)$$

and

$$\begin{Bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \gamma_x \\ \gamma_y \end{Bmatrix} = \begin{bmatrix} [N_1] & [N_0] & [N_{00}] & [N_0] & [N_0] \\ [N_0] & [N_1] & [N_{00}] & [N_0] & [N_0] \\ [N_0] & [N_0] & [N_2] & [N_0] & [N_0] \\ [N_0] & [N_0] & [N_3] & [N_1] & [N_0] \\ [N_0] & [N_0] & [N_4] & [N_0] & [N_1] \\ [N_0] & [N_0] & [N_{00}] & [N_1] & [N_0] \\ [N_0] & [N_0] & [N_{00}] & [N_0] & [N_1] \end{bmatrix} \begin{Bmatrix} \alpha_u \\ \alpha_v \\ \alpha_w \\ \alpha_{\gamma_x} \\ \alpha_{\gamma_y} \end{Bmatrix} = [N_B] \begin{Bmatrix} \alpha_u \\ \alpha_v \\ \alpha_w \\ \alpha_{\gamma_x} \\ \alpha_{\gamma_y} \end{Bmatrix} \text{ for nodes } 4-6 \quad (7)$$

where

$[N_A]$ and $[N_B]$ are matrices of order $8\text{x}45$ and $7\text{x}45$, respectively and $[N_0]$ and $[N_{00}]$ are null matrices of order $1\text{x}6$ and $1\text{x}21$, respectively.

Substituting Eqs. (6) at nodes 1 – 3 and (7) at nodes 4 – 6 the coefficients $(\alpha_i,\ i=1,.....,45)$ in the displacement functions (2) can be expressed in terms of nodal unknowns as

$$\{\delta\} = [A]\{\alpha\} \text{ or } \{\alpha\} = [A]^{-1}\{\delta\}, \quad (8)$$

where

$$[A]^T = \left[[N_A]^{(1)} \ [N_A]^{(2)} \ [N_A]^{(3)} \ [N_B]^{(4)} \ [N_B]^{(5)} \ [N_B]^{(6)} \right], \quad \{\delta\}^T = \{ u_1 \ v_1 \ w_1 \ \theta_{x1} \ \theta_{y1} \ \theta_{xy1} \ \gamma_{x1} \ \gamma_{y1} \ .... \ u_4 \ v_4 \ w_4 \ \theta_{x4} \ \theta_{y4} \ \gamma_{x4} \ \gamma_{y4} \ .... \},$$

$$\{\alpha\}^T = \{ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \alpha_6 \ \alpha_7 \ \alpha_8 \ .... \ \alpha_{45} \},$$

and the matrix $[A]$ having an order of $45 \text{x} 45$ can be formed with (x_i, y_i) and (ξ_i, η_i) of six nodes.

According to first order shear deformation theory (FOSDT), the displacement components of a point at a distance of z from the reference plane may be expressed in terms of field variables $(u, v, w, \gamma_x, \gamma_y)$ at the reference plane as

$$\bar{u}(x,y,z) = u(x,y) - z\left(\frac{\partial w(x,y)}{\partial x} + \gamma_x\right),$$

$$\bar{v}(x,y,z) = v(x,y) - z\left(\frac{\partial w(x,y)}{\partial y} + \gamma_y\right)$$

and $\bar{w}(x,y,z) = w(x,y)$.

The generalized stress-strain relationship may be expressed as

$$\{\sigma\} = [D]\{\varepsilon\}. \tag{9}$$

In the above equation the generalized stress vector is

$$\{\sigma\}^T = \{ N_x \ N_y \ N_{xy} \ M_x \ M_y \ M_{xy} \ Q_x \ Q_y \}. \tag{10}$$

The generalized strain vector $\{\varepsilon\}$ in terms of displacement fields is

$$\{\varepsilon\}^T = \{ \varepsilon_x \ \varepsilon_y \ \varepsilon_{xy} \ \kappa_x \ \kappa_y \ \kappa_{xy} \ \varepsilon_{xz} \ \varepsilon_{yz} \} = \left\{ \frac{\partial u}{\partial x} \ \frac{\partial v}{\partial y} \ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \ -\frac{\partial \theta_x}{\partial x} \ -\frac{\partial \theta_y}{\partial y} \ -\theta_{xy} \ -\gamma_x \ -\gamma_y \right\}, \tag{11}$$

and the rigidity matrix $[D]$ is given by

$$[D] = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ sym. & & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ sym. & & B_{66} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ sym. & & D_{66} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ sym. & & \begin{bmatrix} A_{44} & A_{45} \\ sym. & A_{55} \end{bmatrix} \end{bmatrix}, \tag{12}$$

The rigidity matrix of a laminate $[D]$ is constituted with the contributions of its individual laminae and the material properties $(E_1, E_2, \nu_{12}, G_{12}, G_{13}, G_{23})$ and the fiber orientation of each lamina.

Now, substituting Eqs. (2), (3) and (4) in Eq. (10) the strain-displacement relationship may be expressed as

$$\{\varepsilon\} = [\bar{B}]\{\alpha\} \tag{13}$$

where $[\bar{B}]$ is a $(8 \text{x} 45)$ matrix and is given by

$$[\bar{B}]=\begin{bmatrix}[N_7] & [N_0] & [N_{00}] & [N_0] & [N_0]\\ [N_0] & [N_6] & [N_{00}] & [N_0] & [N_0]\\ [N_6] & [N_7] & [N_{00}] & [N_0] & [N_0]\\ [N_0] & [N_0] & -[N_8] & -[N_7] & [N_0]\\ [N_0] & [N_0] & -[N_9] & [N_0] & -[N_6]\\ [N_0] & [N_0] & -[N_5] & -[N_6] & -[N_7]\\ [N_0] & [N_0] & [N_{00}] & -[N_1] & [N_0]\\ [N_0] & [N_0] & [N_{00}] & [N_0] & -[N_1]\end{bmatrix} \quad (14)$$

where

$[N_8]=\frac{\partial}{\partial x}[N_3]$ and $[N_9]=\frac{\partial}{\partial y}[N_4]$.

Finally, the strain vector $\{\varepsilon\}$ can be expressed as

$$\{\varepsilon\}=[\bar{B}][A]^{-1}\{\delta\} \text{ or } \{\varepsilon\}=[B]\{\delta\} \quad (15)$$

Once the strain displacement matrix $[B]$ and the rigidity matrix $[D]$ are obtained the stiffness matrix can be obtained following the virtual work technique and it may be expressed as

$$[K]^e=\int[B]^T[D][B]\,\mathrm{d}x\,\mathrm{d}y \quad (16)$$

In a similar manner, the consistent mass matrix for an element can be expressed as

$$[M]^e=\int[A]^{-T}[C_1]^T[\bar{\rho}][C_1][A]^{-1}\mathrm{d}x\,\mathrm{d}y, \quad (17)$$

and the equation of motion of an element may be expressed as

$$[K]^e-\omega^2[M]^e=0. \quad (18)$$

where $[C_1]=\begin{bmatrix}[N_1] & [N_0] & [N_{00}] & [N_0] & [N_0]\\ [N_0] & [N_1] & [N_{00}] & [N_0] & [N_0]\\ [N_0] & [N_0] & [N_2] & [N_0] & [N_0]\\ [N_0] & [N_0] & [N_3] & [N_1] & [N_0]\\ [N_0] & [N_0] & [N_4] & [N_0] & [N_1]\end{bmatrix}$

and $[\bar{\rho}]=\begin{bmatrix}\rho h & 0 & 0 & 0 & 0\\ 0 & \rho h & 0 & 0 & 0\\ 0 & 0 & \rho h & 0 & 0\\ 0 & 0 & 0 & \frac{\rho h^3}{12} & 0\\ 0 & 0 & 0 & 0 & \frac{\rho h^3}{12}\end{bmatrix}$.

In the above equation matrices $[C_1]$ and $[\bar{\rho}]$ are of order 5x45 and 5x5, respectively and ρ and h are the density of the plate/laminate material and thickness of the plate/laminate, respectively. Finally, the element stiffness and mass matrices are assembled together to form the overall stiffness matrix $[K]$ and mass matrix $[M]$, respectively.

Integration in the above equations (16) and (17) is carried out using Gauss quadrature method.

3. RESULTS AND DISCUSSIONS

Isotropic uniform thickness rectangular plates having different thickness ratios, aspect ratios and different boundary conditions have been considered. The required Fortran program is developed based on the formulation discussed in the previous section for free vibration analysis of isotropic plates as well as composite plates. Analyses have been performed both on free vibration of isotropic and laminated rectangular plates having different thickness ratios, aspect ratios, boundary conditions, fiber orientation angles and number of layers as applicable. The results of the analyses have been compared with the available published results.

The boundary conditions used in the study are:

a. Boundary line parallel to *x*-axis:

- Simply supported condition: $u = w = \theta_x = \theta_{xy} = \gamma_x = 0$
- Clamped condition: $u = v = w = \theta_x = \theta_y = \theta_{xy} = \gamma_x = \gamma_y = 0$
- Symmetric condition: $v = \theta_y = \theta_{xy} = \gamma_y = 0$.

b. Boundary line parallel to y-axis:

- Simply supported condition: $v = w = \theta_y = \theta_{xy} = \gamma_y = 0$
- Clamped condition: $u = v = w = \theta_x = \theta_y = \theta_{xy} = \gamma_x = \gamma_y = 0$
- Symmetric condition: $u = \theta_x = \theta_{xy} = \gamma_x = 0$.

3.1 Isotropic Rectangular Plates

The Poisson's ratio (ν) and the shear correction factor (κ) of the plate material are taken as 0.3 and 5/6 respectively, unless specified. The eigenvalue tests of the stiffness matrix of a single element having different configurations have been performed and it has been found that the element is free from any spurious modes. Next, the free vibration analysis of rectangular isotropic plates having different boundary conditions is performed. The boundary condition is designated as SCFC if the boundary line parallel to *y*-axis and *x*=0 is simply supported, the boundary line parallel to *y*-axis and *x*=a is clamped, the boundary line parallel to *x*-axis and *y*=0 is free and the boundary line parallel to *x*-axis and *y*=b is clamped, respectively. Mesh division is designated as $(m \times n)$ if the plate is divided into *m* equal divisions along the *x*-direction and *n* equal divisions along the *y*-direction as shown in Fig. 2.

An isotropic rectangular plate is first studied to validate the programme for different mesh divisions, boundary conditions and thickness ratios $(t/b = 0.001 \text{ and } 0.2)$ and aspect ratios $(a/b = 1.0 \text{ and } 2.0)$. The mesh arrangement used is shown in Fig. 2 which is also used in the subsequent problems. The plate is analyzed for different boundary conditions i.e., CFFF,CSCS and CSFF. Thefirstfournon-dimensional frequency parameters

Table 1. *Frequency parameters* $\left(\lambda = \omega b^2 / \pi^2 \sqrt{\rho h / D}\right)$ *for isotropic rectangular plates with different boundary conditions*

a/b	*h/b*	Source	Frequency modes											
			CFFF				CSCS				CSFF			
			1	2	3	4	1	2	3	4	1	2	3	4
1.0	0.001	PS-8	0.346	0.847	2.127	2.706	2.690	5.991	6.065	9.210	1.515	2.048	3.948	4.918
		PS-12	0.346	0.848	2.131	2.709	2.691	5.999	6.074	9.222	1.517	2.052	3.954	4.931
		PS-16	0.346	0.849	2.132	2.710	2.692	6.001	6.076	9.225	1.518	2.054	3.956	4.936
		PS-20	0.346	0.849	2.133	2.711	2.692	6.002	6.077	9.227	1.518	2.055	3.957	4.938
		Liew et al. [11]	**0.354**	**0.863**	**2.157**	**2.756**	**2.741**	**6.133**	**6.159**	**9.406**	**1.540**	**2.086**	**4.026**	**5.011**
	0.2	PS-8	0.334	0.736	1.762	2.242	2.170	4.192	4.254	5.859	1.307	1.680	3.012	3.581
		PS-12	0.333	0.736	1.763	2.243	2.171	4.196	4.258	5.865	1.308	1.681	3.013	3.587
		PS-16	0.333	0.736	1.764	2.244	2.172	4.197	4.260	5.867	1.309	1.682	3.013	3.589
		PS-20	0.333	0.736	1.764	2.244	2.172	4.198	4.261	5.868	1.309	1.682	3.014	3.590
		Liew et al. [11]	**0.338**	**0.744**	**1.780**	**2.276**	**2.202**	**4.259**	**4.297**	**5.934**	**1.325**	**1.701**	**3.052**	**3.626**

2.0	0.001	PS-8	0.086	0.371	0.541	1.209	1.758	2.504	3.784	5.143	0.380	0.786	1.235	1.828
		PS-12	0.087	0.372	0.541	1.210	1.762	2.509	3.793	5.169	0.380	0.786	1.236	1.830
		PS-16	0.087	0.372	0.541	1.211	1.763	2.511	3.797	5.179	0.380	0.787	1.237	1.831
		PS-20	0.087	0.372	0.541	1.211	1.764	2.512	3.799	5.184	0.380	0.787	1.237	1.832
		Liew et al. [11]	**0.096**	**0.377**	**0.544**	**1.221**	**1.800**	**2.553**	**3.847**	**5.303**	**0.383**	**0.793**	**1.243**	**1.845**
	0.2	PS-8	0.085	0.334	0.510	1.053	2.690	5.991	6.065	9.210	0.363	0.697	1.107	1.551
		PS-12	0.085	0.334	0.510	1.053	2.691	5.999	6.074	9.222	0.363	0.697	1.106	1.552
		PS-16	0.085	0.334	0.510	1.053	2.692	6.001	6.076	9.225	0.363	0.697	1.106	1.552
		PS-20	0.085	0.334	0.511	1.053	2.692	6.002	6.077	9.227	0.363	0.697	1.105	1.552
		Liew et al. [11]	**0.085**	**0.336**	**0.512**	**1.059**	**2.741**	**6.133**	**6.159**	**9.406**	**0.364**	**0.701**	**1.115**	**1.560**

$\left(\lambda=\omega b^2/\pi^2\sqrt{\rho t/D}\right)$ obtained by the present model are presented in Table 1 with the results by Liew et al.[11]. Liewet al.[11] has used mathematically complete two-dimensional polynomials in the displacement and rotation functions to approximate the appropriate mode shapes with the Rayleigh-Ritz procedure. The consistent mass matrix is used for the present analysis. Table shows a close agreement between the results.

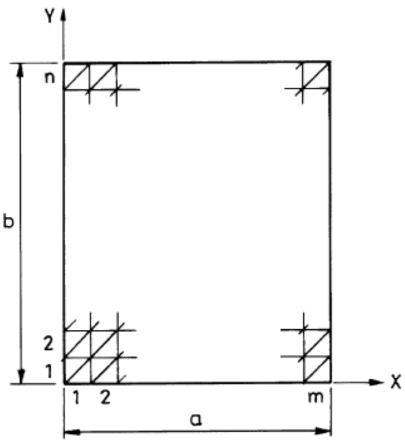

Fig. 2. *Isotropic plate having mesh division of* $m\times n$.

3.2 Composite Rectangular Plates

The problem of free vibration analysis of composite rectangular plates having different lamina layers is considered. The laminate consists of layers of equal thickness. The plate is analyzed by using four different mess sizes i.e. 8×8, 12×12, 16×16 and 20×20. The rectangular plate is analyzed numerically with different boundary conditions, namely SSSS, CCCC and CSFF. with different thickness ratios $(t/b=0.001 \text{ and } 0.2)$ and aspect ratios $(a/b=1.0 \text{ and } 2.0)$. A shear correction factor $k=\pi^2/12$ is used for all the numerical computations. The material properties of each lamina are taken as $E_1/E_2=40$, $G_{23}=0.5E_2$, $G_{12}=G_{31}=0.6E_2$, $\nu_{12}=0.25$, and $\nu_{21}=0.00625$. From the eigenvalue tests of the stiffness matrix of a single element it has also been found that the element does not possess any spurious mode. The first four non-dimensional frequency parameters $\left(\lambda=\omega b^2/\pi^2\sqrt{\rho t/D_0}\right)$ obtained by the present analysis are presented in Table 2 with the results by Liew[12]where $D_0=E_2h^3/12(1-\nu_{12}\nu_{21})$. Liew[12] performed the analysis of free flexural vibration of thick symmetric rectangular laminates using the Ritz method with a set of admissible beam characteristic orthogonal polynomials. Three-ply, five-ply and eight-ply symmetric laminates are studied to investigate the effects of different parameters as mentioned above.Three ply laminated plates having stacking sequence $\left(0^0,90^0,0^0\right)$ are used for the all the numerical computations. The first four non-dimensional frequency parameters are shown in Table 2.Table shows that the present results are very close to the results obtained by Liew[12].

Table 2. *Frequency parameters* $\left(\lambda = \omega b^2 / \pi^2 \sqrt{\rho h / D_0}\right)$ *for three-ply laminated plates* $\left(0^0, 90^0, 0^0\right)$ *with different boundary conditions*

a/b	*h/b*	Source	Frequency modes											
			SSSS				CCCC				CFFF			
			1	2	3	4	1	2	3	4	1	2	3	4
1.0	0.001	PS-8	6.475	9.283	16.027	25.751	14.278	17.207	24.093	35.072	2.152	2.411	4.693	10.483
		PS-12	6.482	9.298	16.054	25.864	14.333	17.281	24.190	35.204	2.158	2.426	4.707	10.496
		PS-16	6.485	9.303	16.063	25.902	14.351	17.306	24.224	35.250	2.161	2.433	4.713	10.501
		PS-20	6.486	9.305	16.068	25.919	14.360	17.318	24.240	35.271	2.162	2.437	4.717	10.504
		Liew[12]	**6.625**	**9.447**	**16.205**	**25.115**	**14.666**	**17.614**	**24.511**	**35.532**	**2.212**	**2.489**	**4.753**	**10.537**
	0.2	PS-8	3.574	5.756	7.407	8.697	4.456	6.659	7.718	9.208	1.417	1.519	3.458	4.652
		PS-12	3.576	5.760	7.410	8.700	4.458	6.661	7.721	9.211	1.420	1.524	3.460	4.658
		PS-16	3.577	5.761	7.411	8.702	4.458	6.662	7.722	9.213	1.421	1.526	3.460	4.662
		PS-20	3.577	5.762	7.411	8.703	4.459	6.663	7.722	9.214	1.422	1.527	3.461	4.663
		Liew [12]	**3.594**	**5.769**	**7.379**	**8.688**	**4.447**	**6.642**	**7.700**	**9.185**	**1.444**	**1.545**	**3.466**	**4.687**
2.0	0.001	PS-8	2.314	6.512	6.551	9.233	5.026	10.347	10.401	14.019	0.546	0.780	3.419	3.695
		PS-12	2.315	6.518	6.558	9.250	5.031	10.370	10.426	14.065	0.547	0.783	3.424	3.698
		PS-16	2.315	6.520	6.560	9.255	5.033	10.377	10.434	14.080	0.546	0.783	3.424	3.700
		PS-20	2.316	6.520	6.561	9.257	5.036	10.380	10.437	14.086	0.547	0.782	3.425	3.700
		Liew [12]	**2.362**	**6.625**	**6.665**	**9.447**	**5.105**	**10.527**	**10.583**	**14.324**	**0.553**	**0.788**	**3.463**	**3.746**
	0.2	PS-8	1.911	3.575	4.821	5.482	3.031	4.239	5.766	5.906	0.474	0.617	1.934	2.121
		PS-12	1.913	3.578	4.827	5.486	3.033	4.242	5.774	5.911	0.474	0.618	1.935	2.122
		PS-16	1.913	3.579	4.830	5.488	3.033	4.244	5.777	5.913	0.474	0.618	1.936	2.123
		PS-20	1.913	3.580	4.831	5.489	3.033	4.244	5.778	5.913	0.474	0.618	1.936	2.123
		Liew [12]	**1.939**	**3.594**	**4.876**	**5.485**	**3.045**	**4.248**	**5.792**	**5.905**	**0.480**	**0.621**	**1.939**	**2.126**

4. CONCLUSIONS

A six-node triangular plate bending element with forty-five degrees of freedom has been used to investigate free vibration of isotropic and composite rectangular plates with different thickness ratios, aspect ratios, boundary conditions, and stacking sequences. Considering different mesh divisions (8x8, 12x12, 16x16 and 20x20) a comparative study of present results with those of earlier investigators shows the convergence characteristics and accuracy of the present element for thin (thickness ratio of 0.001) to thick (thickness ratio of 0.2) plates. It has also been found that the element is free of shear locking and does not exhibit any spurious modes.

REFERENCES

[1] Zienkiewics, O. C., and Tailor, R. L., 'The finite element (two volumes)', New York, McGraw-Hill, 1988.

[2] Timoshenko, S. P., and Gere, J. M., 'Theory of elastic stability', McGraw-Hill International, New York , 2nd Edition, 1963.

[3] Timoshenko, S. P., and Woinowsky-Kriegers, S., 'Theory of plates and Shell', McGrraw-Hill International, New York , 2nd Edition , 1956.

[4] Zienkiewics, O. C., Tailor, R. L., Papadopoulos, P., and Onate, E., 'Plate bending elements with discrete constraints: new triangular elements', Comput. Struct., 35, 505 – 522, 1990.

[5] Chen, W.J., and Cheung, Y.K., 'Refined 9-dof triangular Mindlin plate elements', Int. J. Numer. Methods Eng. 51, 1259 – 1281, 2001.

[6] SitaThankam, V., Gajbir Singh, VenkateswaraRaoG., and Rath, A. K., 'Thermal post-buckling behaviour of laminated plates using a shear flexible element based on coupled-displacement field',Composite Structure , 59, 351-359, 2003.

[7] Shufrin, I., and Eisenberger, M., 'Vibration of shear deformable plates with variable thickness – first-order and higher-order analyses', Journal of Sound and Vibration , 290, 465-489, 2006.

[8] Sang Wook Kang and Sang-Hyun Kim, 'Variation analysis of simply supported rectangular plates with unidirectionally, arbitrarily varying thickness', Journal of Sound and Vibration, 312, 551-562, 2008.

[9] Huang, M., Ma, X. Q., Sakiyama, T., Matuda, H., and Morita, C., 'Free vibration analysis of orthotropic rectangular plates with variable thickness and general boundary conditions', Journal of Sound and Vibration, 228, 931-955, 2005.

[10] Cai, Y.C., Tian. L.G., Atluri, S.N., 'A simple locking-free discrete shear triangular plate element', Comput.Model. Eng. Sci., 77, 221- - 238, 2011.

[11] Liew, K. M., Xiang, Y., and Kitipornchai, S., 'Transeverse vibration of thick rectangular plates – I. Comprehensive set of boundary condition',Comput. Struct., 49, 1-29, 1993.

[12] Liew, K. M., 'Solving the variation thick symmetric laminates by reissner / mindlin plate theory and P-Ritz method', J. Sound Vib., 198(3), 343-360,1996.

[13] Choo, Y.S., Choi, N., and Lee, B.C., 'A new hybrid –Trefftz triangular and quadrilateral plate elements', Appl. Math. Model., 34, 29 – 34, 2010.

[14] Brasile, S., 'An isostatic assumed stress triangular element for Reissner-Mindlin plate-bending problem', Int. J. Numer. Methods Eng., 74, 971 – 995, 2008.

[15] Zhuang, X.Y., Huang, R.Q., Zhu, H.H., Askes, H., and Mathisen, K., 'A new and simple locking-free triangular thick plate element using independent shear degrees of freedom', Finite Elements in Analysis and Design, 75, 1 – 7, 2013.